

On the largest dynamic monopolies of graphs with a given average threshold

Kaveh Khoshkhan Manouchehr Zaker*

Department of Mathematics,
Institute for Advanced Studies in Basic Sciences,
Zanjan 45137-66731, Iran

Abstract

Let G be a graph and τ be an assignment of nonnegative integer thresholds to the vertices of G . A subset of vertices D is said to be a τ -dynamic monopoly, if $V(G)$ can be partitioned into subsets D_0, D_1, \dots, D_k such that $D_0 = D$ and for any $i \in \{0, \dots, k-1\}$, each vertex v in D_{i+1} has at least $\tau(v)$ neighbors in $D_0 \cup \dots \cup D_i$. Denote the size of smallest τ -dynamic monopoly by $\text{dyn}_\tau(G)$ and the average of thresholds in τ by $\bar{\tau}$. We show that the values of $\text{dyn}_\tau(G)$ over all assignments τ with the same average threshold is a continuous set of integers. For any positive number t , denote the maximum $\text{dyn}_\tau(G)$ taken over all threshold assignments τ with $\bar{\tau} \leq t$, by $L\text{dyn}_t(G)$. In fact, $L\text{dyn}_t(G)$ shows the worst-case value of a dynamic monopoly when the average threshold is a given number t . We investigate under what conditions on t , there exists an upper bound for $L\text{dyn}_t(G)$ of the form $c|G|$, where $c < 1$. Next, we show that $L\text{dyn}_t(G)$ is coNP-hard for planar graphs but has polynomial-time solution for forests.

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1 Introduction

In this paper we deal with simple undirected graphs. For any such graph $G = (V, E)$, we denote the cardinality of its vertex set by $|G|$ and the edge density of graph G by

*E-mail: mzaker@iasbs.ac.ir

$\epsilon(G) := |E|/|G|$. We denote the degree of a vertex v in G by $\deg_G(v)$. For other graph theoretical notations we refer the reader to [2]. By a threshold assignment for the vertices of G we mean any function $\tau : V(G) \rightarrow \mathbb{N} \cup \{0\}$. A subset of vertices D is said to be a τ -dynamic monopoly of G or simply τ -dynamo of G , if for some nonnegative integer k , the vertices of G can be partitioned into subsets D_0, D_1, \dots, D_k such that $D_0 = D$ and for any i , $1 \leq i \leq k$, the set D_i consists of all vertices v which has at least $\tau(v)$ neighbors in $D_0 \cup \dots \cup D_{i-1}$. Denote the smallest size of any τ -dynamo of G by $\text{dyn}_\tau(G)$. Dynamic monopolies are in fact modeling the spread of influence in social networks. The spread of innovation or a new product in a community, spread of opinion in Yes-No elections, spread of virus in the internet, spread of disease in a population are some examples of these phenomena. Obviously, if for a vertex v we have $\tau(v) = \deg_G(v) + 1$ then v should belong to any dynamic monopoly of (G, τ) . We call such a vertex v *self-opinioned* (from another interpretation it can be called *vaccinated vertex*). Irreversible dynamic monopolies and the equivalent concepts target set selection and conversion sets have been the subject of active research in recent years by many authors [3, 4, 6, 7, 8, 10, 11, 12, 13].

In this paper by (G, τ) we mean a graph G and a threshold assignment for the vertices of G . The average threshold of τ , denoted by $\bar{\tau}$, is $\sum_{v \in V(G)} \tau(v)/|G|$. In Proposition 1 we show that the values of $\text{dyn}_\tau(G)$ over all threshold assignments with the same average threshold form a continuous set of integers. The maximum element of this set has been studied first time in [10], where the following notation was introduced. Let t be a non-negative rational number such that $t|G|$ is an integer, then $\text{Dyn}_t(G)$ is defined as $\text{Dyn}_t(G) = \max_{\tau: \bar{\tau}=t} \text{dyn}_\tau(G)$. The smallest size of dynamic monopolies with a given average threshold was introduced and studied in [13]. Dynamic monopolies with given average threshold was also recently studied in [5]. In the definition of $\text{Dyn}_t(G)$, it is assumed that $t|G|$ is integer. In order to consider all values of t , we modify a little bit the definition. But we are forced to make a new notation, i.e. $\text{Ldyn}_t(G)$ (which stands for the largest dynamo). The formal definition is as follows.

Definition 1. Let G be a graph and t a positive number. We define $\text{Ldyn}_t(G) = \max\{\text{dyn}_\tau(G) \mid \bar{\tau} \leq t\}$. Assume that a subset $D \subseteq V(G)$ and an assignment of thresholds τ_0 are such that $\bar{\tau}_0 \leq t$, $|D| = \text{dyn}_{\tau_0}(G) = \text{Ldyn}_t(G)$ and D is a τ_0 -dynamic monopoly of (G, τ_0) . Then we say (D, τ_0) is a t -Ldynamo of G .

$\text{Ldyn}_t(G)$ does in fact show the worst-case value of a dynamic monopoly when the average threshold is a prescribed given number. The following concept is motivated by the concept of dynamo-unbounded family of graphs, defined in [12] concerning the smallest size of dynamic monopolies in graphs.

Definition 2. Let for any $n \in \mathbb{N}$, G_n be a graph and t_n be a number such that $0 \leq t_n \leq 2\epsilon(G_n)$. We say $\{(G_n, t_n)\}_{n \in \mathbb{N}}$ is Ldynamo-bounded if there exists a constant $\lambda < 1$ such that for any n , $\text{Ldyn}_{t_n}(G_n) \leq \lambda|G_n|$.

Outline of the paper is as follows. In Section 2, we show that the values of $\text{dyn}_\tau(G)$ over all assignments τ with the same average threshold is a continuous set of integers (Proposition 1). Then we obtain a necessary and sufficient condition for a family of graphs to be Ldynamo-bounded (Propositions 3 and 4). In Section 3, it is shown that the decision problem $\text{Ldynamo}(k)$ (to be defined later) is coNP-hard for planar graphs (Theorem 1) but has polynomial-time solution for forests (Theorem 3).

2 Some results on $\text{Ldyn}_t(G)$

We first show that the values of $\text{dyn}_\tau(G)$ over all threshold assignments τ with the same average threshold are continuous. We need the following lemma from [11].

Lemma 1. [11] *Let G be a graph and τ and τ' be two threshold assignments to the vertices of G such that $\tau(u) = \tau'(u)$ for all vertices u of G except for exactly one vertex, say v . Then*

$$\begin{cases} \text{dyn}_\tau(G) - 1 \leq \text{dyn}_{\tau'}(G) \leq \text{dyn}_\tau(G), & \text{if } \tau(v) > \tau'(v), \\ \text{dyn}_\tau(G) \leq \text{dyn}_{\tau'}(G) \leq \text{dyn}_\tau(G) + 1, & \text{if } \tau(v) < \tau'(v). \end{cases}$$

The continuity result is as follows.

Proposition 1. *Let τ and τ' be two threshold assignments for the vertices of G such that $\bar{\tau} = \bar{\tau}'$. Let also r be an integer such that $\text{dyn}_\tau(G) \leq r \leq \text{dyn}_{\tau'}(G)$. Then there exists τ'' with $\bar{\tau} = \bar{\tau}''$ such that $\text{dyn}_{\tau''}(G) = r$.*

Proof. For any two threshold assignments τ and τ' with the same average threshold, define $\delta(\tau, \tau') = \sum_{v: \tau(v) > \tau'(v)} (\tau(v) - \tau'(v))$. We prove the proposition by the induction on $\delta(\tau, \tau')$. If $\delta(\tau, \tau') = 0$ then for any vertex v , $\tau(v) \leq \tau'(v)$. But the average thresholds are the same, hence $\tau = \tau'$ and the assertion is trivial. Let $k \geq 1$ and assume that the proposition holds for any two τ and τ' with the same average threshold such that $\delta(\tau, \tau') \leq k$. We prove it for $k + 1$. Assume that τ and τ' are given such that $\delta(\tau, \tau') = k + 1$ and $\tau \neq \tau'$. Define $W = \{v : \tau(v) > \tau'(v)\}$. Let $w \in W$. There exists a vertex u such that $\tau(u) < \tau'(u)$. Since otherwise by $\bar{\tau} = \bar{\tau}'$ we would have $\tau = \tau'$. Define a new threshold τ'' as follows. For any vertex v with $v \notin \{u, w\}$ set $\tau''(v) = \tau(v)$. Set also $\tau''(w) = \tau(w) - 1$ and $\tau''(u) = \tau(u) + 1$. We have $\delta(\tau'', \tau') \leq k$, also the average threshold of τ'' is the same as that of τ . So the assertion holds for τ'' and τ' . By Lemma 1 we have $|\text{dyn}_\tau(G) - \text{dyn}_{\tau''}(G)| \leq 1$. We conclude that the assertion holds for τ and τ' too. \square

Let G be a graph and t be a positive number such that $t|G|$ is integer. Let τ be any assignment with average t such that $\tau(v) \leq \deg_G(v)$ for any vertex v . Let $d_1 \leq d_2 \leq$

$\dots \leq d_n$ be a degree sequence of G in increasing form. It was proved in [10] that the size of any τ -dynamic monopoly of G is at most $\max\{k : \sum_{i=1}^k (d_i + 1) \leq nt\}$. The proof of this result in [10] shows that if we allow $\tau(v) = \deg_G(v) + 1$ for some vertices v of G , then the same assertion still holds. We have the following proposition concerning this fact.

Proposition 2. *Let t be a positive number. Assume that in the definition of $Ldyn_t(G)$, the threshold assignments are allowed to have self-opinioned vertices. Then $Ldyn_t(G)$ can be easily obtained by a polynomial-time algorithm.*

Proof. Let $d_1 \leq d_2 \leq \dots \leq d_n$ be a degree sequence of G in increasing form. By the argument we made before Proposition 2, we have $Ldyn_t(G) \leq \max\{k : \sum_{i=1}^k (d_i + 1) \leq nt\}$. Let $k_0 = \max\{k : \sum_{i=1}^k (d_i + 1) \leq nt\}$. We obtain a threshold assignment τ as follows.

$$\tau(v_i) = \begin{cases} \deg_G(v_i) + 1 & i \leq k_0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = \{v_1, v_2, \dots, v_{k_0}\}$. It's clear that (D, τ) is a t -Ldynamo of G .

□

In [10], it was proved that there exists an infinite sequence of graphs G_1, G_2, \dots such that $|G_n| \rightarrow \infty$ and $\lim_{n \rightarrow \infty} Ldyn_{\epsilon(G_n)}(G_n)/|G_n| = 1$. In the following, we show that a stronger result holds. In fact we show that not only the same result holds for $Ldyn_{k\epsilon(G_n)}(G_n)$, where k is any constant with $0 < k \leq 2$, but also it holds for any sequence k_n for which $k_n|G_n| \rightarrow \infty$. In opposite direction, Proposition 4 shows that if $k_n = \mathcal{O}(1/|G_n|)$ then $\lim_{n \rightarrow \infty} Ldyn_{k_n\epsilon(G_n)}(G_n)/|G_n| \neq 1$.

Proposition 3. *There exists an infinite sequence of graphs $\{(G_n, \tau_n)\}_{n=1}^\infty$ satisfying $|G_n| \rightarrow \infty$ and $\epsilon(G_n)/|G_n| = o(\bar{\tau}_n)$ such that*

$$\lim_{n \rightarrow \infty} \frac{Ldyn_{\bar{\tau}}(G_n)}{|G_n|} = 1.$$

Proof. We construct G_n as follows. The vertex set of G_n is disjoint union of a complete graph K_n and n copies of complete graphs K_{n+1} . There exists exactly one edge between each copy of K_{n+1} and K_n . Set $\tau_n(v) = 0$ for each vertex v in K_n and $\tau_n(v) = \deg(v)$ for each vertex v in any copy of K_{n+1} . It is clear that any dynamic monopoly of G_n includes at least n vertices of each copy of K_{n+1} and hence $Ldyn_{\bar{\tau}}(G_n) \geq n^2$. Then we have

$$1 \geq \lim_{n \rightarrow \infty} \frac{Ldyn_{\bar{\tau}}(G_n)}{|G_n|} \geq \lim_{n \rightarrow \infty} \frac{n^2}{n(n+2)} = \lim_{n \rightarrow \infty} \frac{n}{n+2} = 1.$$

To complete the proof we show that $\frac{\bar{\tau}_n}{|E(G_n)|/|V(G_n)|^2} \rightarrow \infty$.

$$\lim_{n \rightarrow \infty} \frac{\bar{\tau}_n}{|E(G_n)|/|V(G_n)|^2} = \lim_{n \rightarrow \infty} \frac{(n^2 + n + 1)/(n + 2)}{(n^2 + n + n(n + n^2))/2(n^2 + 2n)^2} = \infty.$$

□

Proposition 3 shows that if t_n is such that $\epsilon(G_n)/|G_n| = o(t_n)$ then $\{(G_n, t_n)\}_n$ is not necessarily Ldynamo-bounded. In opposite direction, the next proposition shows that if there exists a positive number c such that t_n satisfies $t_n \leq c\epsilon(G_n)/|G_n|$, then any family $\{(G_n, t_n)\}_n$ is Ldynamo-bounded.

Proposition 4. *Let G be a graph and c and t be two constants such that $t \leq c \frac{\epsilon(G)}{|G|}$.*

Then

$$Ldyn_t(G) < \frac{c}{c+1}|G|.$$

Proof. Let n be the order of G . If $n < c/2$, then $\lceil cn/(c+1) \rceil = n$ and hence the inequality $Ldyn_t(G) < c|G|/(c+1)$ is trivial. Assume now that $n \geq c/2$. Let $d_1 \leq d_2 \leq \dots \leq d_n$ be a degree sequence of G in increasing form and set $k_0 = \max\{k : \sum_{i=1}^k (d_i + 1) \leq nt\}$. As we mentioned before, by a result from [10] we have $Ldyn_t(G) \leq k_0$. The assumption $t \leq c(\epsilon(G)/n)$ implies $nt \leq (c/2n) \sum_{i=1}^n d_i$ and hence $\sum_{i=1}^{k_0} (d_i + 1) \leq (c/2n) \sum_{i=1}^n d_i$ or equivalently $(2n/c) \leq (\sum_{i=1}^n d_i) / \sum_{i=1}^{k_0} (d_i + 1)$. Assume on the contrary that $k_0 \geq cn/(c+1)$. Then

$$\frac{2n}{c} \leq \frac{\sum_{i=1}^{k_0} d_i + \sum_{i=k_0+1}^n d_i}{(\sum_{i=1}^{k_0} d_i) + \frac{c}{c+1}n} \leq \frac{(\sum_{i=1}^{k_0} d_i) + \frac{n^2}{c+1}}{(\sum_{i=1}^{k_0} d_i) + \frac{c}{c+1}n}.$$

Therefore

$$\frac{2n-c}{c} \sum_{i=1}^{k_0} d_i \leq \frac{n^2}{c+1} - \frac{2n^2}{c+1}.$$

The left hand side of the last inequality is positive but the other side is negative. This contradiction implies $k_0 < cn/(c+1)$, as required. □

3 Algorithmic results

Algorithmic results concerning determining $dyn_\tau(G)$, with various types of threshold assignments such as constant thresholds or majority thresholds, were studied in

[4, 6, 7]. In this section, we first show that it is a coNP-hard problem on planar graphs to compute the size of D such that (D, τ) is a $k\epsilon(G)$ -Ldynamo of G . Then we prove that the same problem has a polynomial-time solution for forests. The formal definition of the decision problem concerning Ldynamo is the following, where k is any arbitrary but fixed real number with $0 < k \leq 2$.

Name: LARGEST DYNAMIC MONOPOLY (Ldynamo(k))

Instance: A graph G on say n vertices and a positive integer d .

Question: Is there an assignment of thresholds τ to the vertices of G with $n\bar{\tau} = \lfloor nk\epsilon(G) \rfloor$ such that $\text{dyn}_\tau(G) \geq d$?

The following theorem shows coNP-hardness of the above problem. Recall that Vertex Cover (VC) asks for the smallest number of vertices S in a graph G such that S covers any edge of G . Denote the smallest cardinality of any vertex cover of G by $\beta(G)$. The problem VC is NP-complete for planar graphs [9].

Theorem 1. *For any fixed k , where $0 < k \leq 2$, Ldynamo(k) is coNP-hard even for planar graphs.*

Proof. We make a polynomial time reduction from VC (planar) to our problem. Let $\langle G, l \rangle$ be an instance of VC, where G is planar. Define $s = 4|E(G)| \times \max\{1, 1/k\} + 14$ and set $p = \lfloor (ks - 2)/(2 - k) \rfloor - |E(G)|$. Construct a graph H from G as follows. To each vertex v of G attach a star graph $K_{1,s-1}$ in such a way that v is connected to the central vertex of the star graph. Consider one of these star graphs and let y be a vertex of degree one in it. Add a path P of length $p - 1$ starting from y (see Figure 1). The path P intersects the rest of the graph only in y . Call the resulting graph H . Since G is planar, H is planar too.

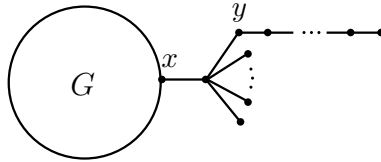


Figure 1: graph H

We claim that $\langle G, l \rangle$ is a yes-instance of VC if and only if $\langle H, l + \lfloor p/2 \rfloor + 1 \rangle$ is a no-instance of Ldynamo(k). From the construction of H , we have $|E(H)| = |E(G)| + s + p$. Then since $p = \lfloor (ks - 2)/(2 - k) \rfloor - |E(G)|$ we have

$$\begin{aligned} p &\leq (ks - 2)/(2 - k) - |E(G)| \\ \Rightarrow 2p + 2|E(G)| + 2 &\leq k(s + p + |E(G)|) \\ \Rightarrow 2p + 2|E(G)| + 2 &\leq \lfloor k|E(H)| \rfloor. \end{aligned}$$

Also from the value of p we have

$$\begin{aligned}
p &\geq (ks - 2)/(2 - k) - |E(G)| - 1 \\
\Rightarrow 2p + 2|E(G)| + 2 + (2 - k) &> k(s + p + |E(G)|) \\
\Rightarrow 2p + 2|E(G)| + 2 + \lfloor 2 - k \rfloor &\geq \lfloor k|E(H)| \rfloor \\
\Rightarrow 2p + 2|E(G)| + 3 &\geq \lfloor k|E(H)| \rfloor.
\end{aligned}$$

Assume first that $\langle G, l \rangle$ is a no-instance of VC. Then $\beta(G) \geq l + 1$. We construct a threshold assignment τ for graph H as follows.

$$\tau(v) = \begin{cases} \deg_H(v) & v \in G \cup P, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

It is easily seen that $\bar{\tau} \leq k\epsilon(H)$ and also $\text{dyn}_{\bar{\tau}}(H) = \beta(G) + \lfloor p/2 \rfloor$. Therefore $\langle H, l + \lfloor p/2 \rfloor + 1 \rangle$ is a yes-instance for $L\text{dynamo}(k)$.

Let $\langle G, l \rangle$ be a yes-instance of VC. Then $\beta(G) < l + 1$. Assume that (D, τ) is a $(k\epsilon(H))$ -Ldynamo of H . The assumption $s > 4|E(G)| + 14$ implies $|D \cap (H \setminus G)| \leq \lfloor p/2 \rfloor$. From the other hand, $|D \cap G| \leq \beta(G) < l + 1$. Hence $|D| < l + \lfloor p/2 \rfloor + 1$. This shows that $\langle H, l + \lfloor p/2 \rfloor + 1 \rangle$ is a no-instance for $L\text{dynamo}(k)$. This completes the proof. \square

In the rest of this section we obtain a polynomial-time solution for forests (Theorem 3). We need some prerequisites. We will make use of the concept of resistant subgraphs, defined in [12] as follows. Given (G, τ) , any induced subgraph $K \subseteq G$ is said to be a τ -resistant subgraph in G , if for any vertex $v \in K$ the inequality $\deg_K(v) \geq \deg_G(v) - \tau(v) + 1$ holds, where $\deg_G(v)$ is the degree of v in G . The following proposition in [12] shows the relation between resistant subgraphs and dynamic monopolies.

Proposition 5. ([12]) *A set $D \subseteq G$ is a τ -dynamo of graph G if and only if $G \setminus D$ does not contain any resistant subgraph.*

The following lemma provides more information on resistant subgraphs which are also triangle-free.

Lemma 2. *Assume that (G, τ) is given. Let also H be a triangle-free τ -resistant subgraph in G and $e = uv$ be any arbitrary edge with $u, v \in H$. Let τ' be defined as follows*

$$\tau'(w) = \begin{cases} \tau(w) & \text{if } w \notin H, \\ 0 & \text{if } w \in H \setminus \{u, v\}, \\ \deg_G(v) & \text{if } w = v, \\ \deg_G(u) & \text{if } w = u, \end{cases}$$

Then $\overline{\tau'} \leq \overline{\tau}$.

Proof. Since H is triangle-free, then $|H| \geq \deg_H(u) + \deg_H(v)$. From the definition of the resistant subgraphs, for any vertex $w \in H$, one has $\tau(w) \geq \deg_{G \setminus H}(w) + 1$. Hence the following inequalities hold.

$$\begin{aligned}
\sum_{w \in H} \tau(w) &\geq \sum_{w \in H} (\deg_{G \setminus H}(w) + 1) \\
&\geq |H| + \deg_{G \setminus H}(u) + \deg_{G \setminus H}(v) \\
&\geq \deg_H(u) + \deg_H(v) + \deg_{G \setminus H}(u) + \deg_{G \setminus H}(v) \\
&= \deg_G(u) + \deg_G(v).
\end{aligned}$$

It turns out that $\sum_{w \in G} \tau'(w) \leq \sum_{w \in G} \tau(w)$ and hence $\overline{\tau'} \leq \overline{\tau}$. \square

By a *(zero, degree)-assignment* we mean any threshold assignment τ for the vertices of a graph G such that for each vertex $v \in V(G)$, either $\tau(v) = 0$ or $\tau(v) = \deg_G(v)$. The following remark is useful and easy to prove. We omit its proof.

Remark 1. Assume that (G, τ) is given where τ is *(zero, degree)-assignment*. Let G_1 be the subgraph of G induced on $\{v \in G \mid \tau(v) = \deg_G(v)\}$. Then every minimum vertex cover of G_1 is a minimum τ -dynamo of G , and vice versa.

The following theorem concerning *(zero, degree)-assignments* in forests is essential in obtaining an algorithm for t -Ldynamo of forests for a given t .

Theorem 2. Let F be a forest and t be a positive constant. There exists a *(zero, degree)-assignment* τ' such that $\overline{\tau'} \leq t$ and

$$Ldyn_t(F) = dyn_{\tau'}(F).$$

Proof. Let (D, τ) be a t -Ldynamo of F . We prove the theorem by induction on $|D|$. Assume first that $|D| = 1$. Then by Proposition 5, F has at least one τ -resistant subgraph say, F' . Let u and v be two adjacent vertices in F' . Let τ' be the threshold assignment constructed in Lemma 2 such that $\tau'(u) = \deg_F(u)$ and $\tau'(v) = \deg_F(v)$. Modify τ' so that $\tau'(w) = 0$ for every vertex $w \in F \setminus \{u, v\}$. It is clear that τ' is a *(zero, degree)-assignment*. The edge uv is a τ' -resistant subgraph in F and hence $dyn_{\tau'}(F) = Ldyn_t(F) = 1$. This proves the induction assertion in this case.

Now assume that the assertion holds for any forest F with $|D| < k$. Let F be a forest with $Ldyn_t(F) = k$ and D be a t -Ldynamo of F with $|D| = k$. Let also F_1 be the largest τ -resistant subgraph of F . For any $v \in F_1$, set $\varphi(v) = \tau(v) - \deg_{F \setminus F_1}(v)$. By the definition of resistant subgraphs, $\varphi(v) > 0$. It is clear that $dyn_{\varphi}(F_1) = k$.

We show that there exists a (zero,degree)-assignment τ'_1 for F_1 such that (D_1, τ'_1) is a $\overline{\varphi}$ -Ldynamo of F_1 with $|D_1| = k$.

Let T be a connected component of F_1 . Consider T as a top-down tree, where the toppest vertex is considered as the root of T . Since T is a φ -resistant subgraph in F_1 , it implies that $D_1 \cap T$ is not the empty set. We argue that D_1 can be chosen in such a way that it does not contain any vertex $w \in T$ with $\varphi(w) = 1$, except possibly the root. The reason is that if $w \in D_1 \cap T$ with $\varphi(w) = 1$, then we replace w by its nearest ancestor (with respect to the root of T) whose threshold is not 1; and if there is no such ancestor then we replace w by the root. Let $v \in D_1 \cap T$ be the farthest vertex from the root of T . Let T_v be the subtree of T consisting of v and its descendants. Obviously $T_v \cap D_1 = \{v\}$.

Now we show that T_v is a φ -resistant subgraph in F_1 . For each vertex $w \in T_v \setminus \{v\}$, since $\varphi(w) \geq 1$ and $\deg_{F_1 \setminus T_v}(w) = 0$, then $\varphi(w) \geq \deg_{F_1 \setminus T_v}(w) + 1$. We have also $\varphi(v) \geq \deg_{F_1 \setminus T_v}(v) + 1$. Since if $\varphi(v) = 1$, then v is the root of T and $T_v = T$ and hence $\deg_{F_1 \setminus T_v}(v) = 0$. And if $\varphi(v) > 1$, then $\deg_{F_1 \setminus T_v}(v) \leq 1$. This proves that T_v is a φ -resistant subgraph in F_1 . Let v' be an arbitrary neighbor of v in T_v . We construct the threshold assignment τ_1 for F_1 as follows.

$$\tau_1(w) = \begin{cases} \varphi(w) & \text{if } w \notin T_v, \\ 0 & \text{if } w \in T_v \setminus \{v, v'\}, \\ \deg_{F_1}(w) & \text{if } w \in \{v, v'\}. \end{cases}$$

By Lemma 2, we have $\overline{\tau_1} \leq \overline{\varphi}$. Since edge vv' is a τ_1 -resistant subgraph in F_1 , then $\text{dyn}_{\tau_1}(F_1) = \text{dyn}_{\varphi}(F_1) = k$ and so D_1 is a minimum τ_1 -dynamo of F_1 . Set $F_2 = F_1 \setminus T_v$. Let u be the parent of the vertex v . Construct the threshold assignment τ_2 for F_2 as follows.

$$\tau_2(w) = \begin{cases} \tau_1(w) & \text{if } w \in F_2 \setminus \{u\}, \\ \tau_1(w) - 1 & \text{if } w = u. \end{cases}$$

It is easily seen that the union of any τ_2 -dynamo of F_2 and $\{v\}$ is a τ_1 -dynamo of F_1 and also $D_1 \setminus \{v\}$ is a τ_2 -dynamo of F_2 . Hence $\text{dyn}_{\tau_2}(F_2) = \text{dyn}_{\tau_1}(F_1) - 1 = k - 1$. Let φ_2 be any threshold assignment for F_2 with $\overline{\varphi_2} = \overline{\tau_2}$. Now construct the threshold assignment φ_1 for F_1 as follows.

$$\varphi_1(w) = \begin{cases} \varphi_2(w) & \text{if } w \in F_2 \setminus \{u\}, \\ \tau_1(w) & \text{if } w \in T_v, \\ \varphi_2(w) + 1 & \text{if } w = u. \end{cases}$$

Because the union of any φ_2 -dynamo of F_2 and $\{v\}$, forms a φ_1 -dynamo of F_1 and also for any φ_1 -dynamo P of F_1 , the set $P \cap F_2$ is a φ_2 -dynamo of F_2 then $P \not\subseteq F_2$.

This result and $\text{dyn}_{\tau_2}(F_2) = k - 1$ imply that $L\text{dyn}_{\bar{\tau}_2}(F_2) = k - 1$. From the induction hypothesis there exists a (zero,degree)-assignment τ'_2 for F_2 with $\bar{\tau}'_2 \leq \bar{\tau}_2$ such that $\text{dyn}_{\tau'_2}(F_2) = k - 1$. Now we construct the (zero,degree)-assignment τ'_1 for F_1 as follows.

$$\tau'_1(w) = \begin{cases} \tau'_2(w) & \text{if } w \in F_2 \setminus \{u\}, \\ \tau_1(w) & \text{if } w \in T_v, \\ \tau'_2(w) + 1 & \text{if } w = u \text{ and } \tau'_2(u) \neq 0 \\ 0 & \text{if } w = u \text{ and } \tau'_2(u) = 0. \end{cases}$$

It is easily seen that $\text{dyn}_{\tau'_1}(F_1) = k$. We finally obtain the desired (zero,degree)-assignment τ' for F as follows.

$$\tau'(w) = \begin{cases} \deg_F(w) & \text{if } w \in F_1, \tau'_1(w) = \deg_{F \setminus F_1}(w), \\ 0 & \text{if } w \in F_1, \tau'_1(w) = 0, \\ 0 & \text{if } w \notin F_1. \end{cases}$$

□

In the following we show that for any forest there exists a (zero,degree)-assignment which is zero outside the vertices of a matching.

Proposition 6. *Let F be a forest and t a positive constant. Then there exists a matching M such that for the (zero,degree)-assignment τ defined below, we have $\bar{\tau} \leq t$ and $L\text{dyn}_t(F) = \text{dyn}_\tau(F) = |M|$,*

$$\tau(w) = \begin{cases} \deg_F(w) & \text{if } w \text{ is a vertex saturated by } M, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2, there exists a (zero,degree)-assignment τ' such that $\bar{\tau}' \leq t$ and $L\text{dyn}_t(F) = \text{dyn}_{\tau'}(F)$. Let F_1 be a subgraph induced on all vertices w , with $\tau'(w) = \deg_F(w)$. Let D be a minimum vertex cover of F_1 . Remark 1 implies that D is a minimum τ' -dynamic monopoly of F . Assume that M is a maximum matching of F_1 . We show that M satisfies the conditions of the theorem. Each edge of M forms a τ -resistant subgraph in F . Hence $\text{dyn}_\tau(F) \geq |M|$. Using the so-called König Theorem on bipartite graphs we have $|D| = |M|$. Consequently, $\text{dyn}_\tau(F) \geq |D| = \text{dyn}_{\tau'}(F) = L\text{dyn}_t(F)$. It is easily seen that $\bar{\tau} \leq \bar{\tau}' \leq t$. The proof completes. □

To prove Theorem 3, we need the following proposition whose proof is given in the appendix.

Proposition 7. *Let G be a bipartite graph, where each edge e has a cost $c(e) \geq 0$. Let also d be a positive number. Then there is a polynomial time algorithm which finds a maximum matching M in G with $\text{cost}(M) \leq d$, where $\text{cost}(M) = \sum_{e \in M} c(e)$.*

We are ready now to present the next result.

Theorem 3. *Given a forest F and a positive number t , there exists an algorithm which computes $Ldyn_t(F)$ in polynomial-time.*

Proof. For each edge $e = uv$ of F define $cost(e) = deg_F(u) + deg_F(v)$ and for each $S \subseteq E(F)$ define $cost(S) = \sum_{e \in S} cost(e)$. Let M be any arbitrary matching and τ be a (zero,degree)-assignment constructed from M as obtained in Proposition 6. It is easily seen that $\bar{\tau} \leq t$ if and only if $cost(M) \leq t|F|$. Now, if M is a maximum matching satisfying $cost(M) \leq t|F|$, then Proposition 6 implies $Ldyn_t(F) = dyn_\tau(F) = |M|$. By Proposition 7 there is a polynomial-time algorithm which finds maximum matching M in F with $cost(M) \leq c$ for any value c . Then using Proposition 6 for given forest F and constant t , there is a polynomial time algorithm which finds a (zero,degree)-assignment τ such that $Ldyn_t(F) = dyn_\tau(F)$. From the other side, finding a minimum vertex cover in bipartite graphs is a polynomial-time problem. Therefore using Remark 1 a minimum τ -dynamic monopoly for F can be found in polynomial-time. \square

For further researches, it would be interesting to obtain other families of graphs for which $Ldynamo(k)$ has polynomial-time solution. Also we don't know yet whether $Ldynamo(k) \in NP \cup coNP$. We guess this is not true.

4 Appendix

We prove Proposition 7 using the minimum cost flow algorithm. The minimum cost flow problem (MCFP) is as follows (see e.g. [1] for details).

Let $G = (V, E)$ be a directed network with a cost $c(i, j) \geq 0$ for any of its edges (i, j) . Also for any edge $(i, j) \in E$ there exists a capacity $u(i, j) \geq 0$. We associate with each vertex $i \in V$ a number $b(i)$ which indicates its source or sink depending on whether $b(i) > 0$ or $b(i) < 0$. The minimum cost flow problem (MCFP) requires the determination of a flow mapping $f : E \rightarrow \mathbb{R}$ with minimum cost $z(f) = \sum_{(i,j) \in E} c(i, j)f(i, j)$ subject to the following two conditions:

- (1) $0 \leq f(i, j) \leq u(i, j)$ for all $(i, j) \in E$ (capacity restriction);
- (2) $\sum_{\{j:(i,j) \in E\}} f(i, j) - \sum_{\{j:(j,i) \in E\}} f(j, i) = b(i)$ for all $i \in V$ (demand restriction).

In [1], a polynomial-time algorithm is given such that determines if such a mapping f exists. And in case of existence, the algorithm outputs f . Furthermore, if all values $u(i, j)$ and $b(i)$ are integers then the algorithm obtains an integer-valued mapping f . In the following we prove Proposition 7.

Theorem. Let $G[X, Y]$ be a bipartite graph with $\text{cost}(ij) \geq 0$ for each edge $ij \in G$ and d be a positive number. Then there exists a polynomial-time algorithm which finds maximum matching M in G with $\text{cost}(M) \leq d$.

Proof. Construct a directed network H from bipartite graph $G[X, Y]$ as follows. Add two new vertices s and t as the source and the sink of H , respectively and directed edges (s, x) for each $x \in X$ and (y, t) for each $y \in Y$. Make all other edges directed from X to Y . For each edge (i, j) set $u(i, j) = 1$ and define $c(i, j)$ as follows.

$$c(i, j) = \begin{cases} 0 & i = s \text{ or } j = t, \\ \text{cost}(ij) & i \in X, j \in Y. \end{cases}$$

For each vertex $i \in X \cup Y$, set $b(i) = 0$ and define $b(s) = -b(t) = k$, where k is an arbitrary positive integer. We have now an instance of MCFP. Assume that there exists a minimum cost flow mapping for this instance (obtained by the above-mentioned algorithm of [1]). Since $u(i, j)$ and $b(i)$ are integers then f is an integer-valued mapping. Therefore $f(i, j)$ is either 0 or 1. Let M be the set of edges (i, j) with $f(i, j) = 1$, where $i \in X$ and $j \in Y$. Clearly M is a matching of size k having $\text{cost}(M) = z(f)$, where $z(f)$ is as defined in MCFP above.

Conversely, let M' be any arbitrary matching in G with $|M'| = k$. We construct a flow mapping f as follows.

$$f(i, j) = \begin{cases} 1 & i \in X, j \in Y, ij \in M', \\ 1 & i = s, jl \in M' \text{ for some } l \in Y, \\ 1 & j = t, li \in M' \text{ for some } l \in X, \\ 0 & \text{otherwise.} \end{cases}$$

The conditions of MCFP are satisfied for f . Also $z(f) = \text{cost}(M')$. We conclude that to obtain a matching of size k with the minimum cost is equivalent to obtain a minimum cost flow mapping for the associated MCFP instance (note that k is a parameter of this instance). We conclude that in order to find a matching M satisfying $\text{cost}(M) \leq d$ and with the maximum size, it is enough to run the corresponding algorithm for the above-constructed MCFP instance for each k , where k varies from 1 to $|G|/2$. Note that $|G|/2$ is an upper bound for the size of any matching. This completes the proof. \square

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